J. Fluid Mech. (1984), *vol.* 146, *pp.* 303–312
Printed in Great Britain

Mass transport in a turbulent boundary layer under a progressive water wave

By S. J. JACOBS

Department of Atmospheric and Oceanic Science, and Department of Mechanical Engineering and Applied Mechanics, The University of Michigan, Ann Arbor, MI 48107

(Received 13 January 1984)

The bottom boundary layer under a progressive water wave is studied using Saffman's turbulence model. Saffman's equations are analysed asymptotically for the case $Re \gg 1$, where Re is a Reynolds number based on a characteristic magnitude of the orbital velocity and a characteristic orbital displacement. Approximate solutions for the mass-transport velocity at the edge of the boundary layer and for the bottom stress are obtained, and Taylor's formula for the rate of energy dissipation is verified. The theoretical results are found to agree well with observations for sufficiently large Reynolds numbers.

1. Introduction

According to Stokes' theory of progressive water waves, the horizontal velocity U at the bottom boundary of a fluid of constant mean depth h is given by

$$U = V[\cos(kx-\sigma t)+\epsilon(c_1+c_2\cos 2(kx-\sigma t))+O(\epsilon^2)], \tag{1.1}$$

where

$$V=\frac{A\sigma}{\sinh kh},\quad \epsilon=\frac{Ak}{\sinh kh},\quad c_2=\frac{\frac{3}{4}}{\sinh^2 kh},\quad \sigma=(gk\tanh kh)^{\frac{1}{2}}, \tag{1.2}$$

in which A is the wave amplitude and c_1 is a dimensionless constant. The force generating steady streaming motions arises in the bottom boundary layer rather than in the main body of the fluid (Lighthill 1978, pp. 337–351), and therefore the constant c_1 in (1.1) and in the expression

$$U_{\mathrm{m}} = (c_1+\tfrac{1}{2})\frac{A^2k\sigma}{\sinh^2 kh} \tag{1.3}$$

for the mass-transport velocity at the bottom boundary must be calculated by solving the equations governing the flow in the boundary layer.

Theory (Longuet-Higgins 1953) and observations (Russell & Osorio 1958; Collins 1963) indicate that $c_1 = \frac{3}{4}$ in the case of laminar flow, and a simple model for the eddy viscosity used by Longuet-Higgins in an appendix to the paper by Russell & Osorio suggests that c_1 takes the same value if the flow in the boundary layer is turbulent. However, the observational papers just cited and subsequent experiments (Brebner, Askew & Law 1966; Bijker, Kalwijk & Pieters 1974) show that the mass-transport velocity at the edge of a turbulent boundary layer is smaller than the value calculated by Longuet-Higgins by about a factor of two, and that the discrepancy between theory and experiment increases with decreasing depth. Furthermore, in some experiments negative mass-transport velocities are observed under breaking waves and in water of shallow depth.

Measurements of the bed shear stress (Riedel, Kamphuis & Brebner 1972) imply that the flow in the boundary layer is turbulent if the Reynolds number $Re \geqslant 10^4$, where Re is given in terms of V, σ, and the kinematic viscosity ν by

$$Re = \frac{V^2}{\nu\sigma}. \tag{1.4}$$

The Reynolds number is $O(10^5)$ or larger for wind waves or swell in water of moderate or small depth, and the flow in the bottom boundary layer under such waves is almost certainly turbulent. Since this flow is not well predicted by Longuet-Higgins' theory, an improved model is needed for application to cases of interest in near-shore oceanography and coastal engineering.

The method used in this paper to calculate the mass transport velocity is based on the approximate equation

$$\langle (U-c)\,\tau\rangle = 0, \tag{1.5}$$

in which C is the phase velocity of the waves, τ is the kinematic bottom stress, and the angle brackets denote the average over a wavelength. Equation (1.5) is implied by Taylor's (1919) hypothesis that the rate of energy dissipation per unit horizontal area is given by $\rho\langle U\tau\rangle$, where ρ is the fluid density. Since momentum is lost at a rate $\rho\langle\tau\rangle$ per unit area and since the ratio of the wave energy to the momentum is equal to the phase velocity, the rate of energy dissipation is also given by $\rho C\langle\tau\rangle$. Equating the two expressions for the rate of energy dissipation yields (1.5), and an equation for c_1 is obtained by substition from (1.1).

Equation (1.5) can be derived without using Taylor's hypothesis by showing that the Kármán integral equation is approximated closely by

$$\tau = \frac{\partial}{\partial t}(U\delta) + \frac{\partial}{\partial x}(U^2\delta) + U\delta\frac{\partial U}{\partial x}, \tag{1.6}$$

where δ is the displacement thickness, and an explicit form for (1.5) can be determined by deriving an expression for the drag cofficient

$$c_{\mathrm{f}} = \frac{2\tau}{|U|\,U}. \tag{1.7}$$

In the present study Saffman's turbulence model (Saffman 1970, 1974; Saffman & Wilcox 1974) is used to carry out this calculation and thus to obtain an approximation for the mass-transport velocity at the edge of the bottom boundary layer. In addition, we will provide a derivation of Taylor's formula for the rate of energy dissipation due to bottom friction. A comparison of calculated and measured mass-transport velocities shows a reasonably good degree of agreement between theory and observations. The expression obtained here for the drag coefficient is also compared with measurements, and shows a similar degree of agreement.

2. Formulation

Let (x, y) denote rectangular coordinates and (u, v) the corresponding Reynolds-averaged velocity components, where $y = 0$ denotes the bottom boundary. In the boundary-layer version of Saffman's model the only significant Reynolds stress is given by

$$\tau_{xy} = \frac{a^2 e}{\omega}\frac{\partial u}{\partial y}, \tag{2.1}$$

where τ_{xy} and e denote the kinematic Reynolds stress and turbulent kinetic energy, a is a constant, and ω is a quantity with the dimensions of a frequency which Saffman calls the pseudovorticity. Saffman's equations can be expressed in the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.2}$$

$$\frac{\mathrm{D}u}{\mathrm{D}t}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\frac{\partial}{\partial y}\left[\left(\frac{a^2e}{\omega}+\nu\right)\frac{\partial u}{\partial y}\right], \tag{2.3}$$

$$\frac{\mathrm{D}e}{\mathrm{D}t}=e\left[a\left|\frac{\partial u}{\partial y}\right|-\omega\right]+\frac{\partial}{\partial y}\left[\left(\frac{a^2e}{2\omega}+\nu\right)\frac{\partial e}{\partial y}\right], \tag{2.4}$$

$$\frac{\mathrm{D}\omega^2}{\mathrm{D}t}=\omega^2\left[b\left|\frac{\partial u}{\partial y}\right|-\frac{c\omega}{a}\right]+\frac{\partial}{\partial y}\left[\left(\frac{a^2e}{2\omega}+\nu\right)\frac{\partial \omega^2}{\partial y}\right], \tag{2.5}$$

where U is given by (1.1) and where

$$\frac{\mathrm{D}}{\mathrm{D}t}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}. \tag{2.6}$$

The values of the constants a, b, and c are

$$a=0.3, \quad b=0.15, \quad c=0.5, \tag{2.7a}$$

and the theory predicts a value

$$\kappa=\left[\frac{c-b}{2}\right]^{\frac{1}{2}}=0.4183 \tag{2.7b}$$

for the Kármán constant. The boundary conditions are

$$u\to U, \quad e\to 0, \quad \omega\to 0 \quad \text{as} \quad y\to\infty, \tag{2.8a}$$

$$u=v=e=0, \quad \omega=a\left|\frac{\partial u}{\partial y}\right|S \quad \text{at} \quad y=0, \tag{2.8b}$$

where S is a function of the roughness length calculated in the paper by Saffman & Wilcox, and the kinematic viscous stress at the boundary is given by

$$\tau=\nu\frac{\partial u}{\partial y} \quad \text{at} \quad y=0. \tag{2.9}$$

It is convenient now to introduce the dimensionless variables

$$\left.\begin{aligned} x'=kx, \quad y'=\frac{ky}{\Delta}, \quad t'=\sigma t, \quad u'=\frac{u}{V}, \quad v'=\frac{v}{V\Delta}, \\ U'=\frac{U}{V}, \quad e'=\frac{k^2ae}{(\sigma\Delta)^2}, \quad \omega'=\frac{\omega}{a\sigma}, \quad \tau'=\frac{k^2\tau}{(\sigma\Delta)^2}, \end{aligned}\right\} \tag{2.10}$$

and the parameters

$$R=\frac{\sigma\Delta^2}{\nu k^2}, \quad \epsilon=\frac{kV}{\sigma}, \quad \beta=\frac{\sigma\Delta}{kV}, \tag{2.11}$$

where ϵ is given by (1.2) and where Δ denotes the dimensionless boundary-layer thickness. This is determined by using the definitions to show that $\Delta = \epsilon\beta$ and by solving

$$\beta = \frac{\kappa}{\log R} = \frac{\kappa}{\log(\beta^2 Re)} \tag{2.12}$$

for β, where Re is defined by (1.4).

In the remainder of this paper the primes on the dimensionless variables will be omitted and dimensional quantities except for the reference and phase velocities V and C will be denoted by an asterisk. The dimensionless version of Saffman's equations is then given by

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{2.13}$$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial U}{\partial t}+\epsilon U\frac{\partial U}{\partial x}+\frac{\partial}{\partial y}\left[\left(\frac{e}{\omega}+\frac{1}{R}\right)\frac{\partial u}{\partial y}\right], \tag{2.14}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = ae\left[\frac{1}{\beta}\left|\frac{\partial u}{\partial y}\right|-\omega\right]+\frac{\partial}{\partial y}\left[\left(\frac{e}{2\omega}+\frac{1}{R}\right)\frac{\partial e}{\partial y}\right], \tag{2.15}$$

$$\frac{\mathrm{d}\omega^2}{\mathrm{d}t} = \omega^2\left[\frac{b}{\beta}\left|\frac{\partial u}{\partial y}\right|-c\omega\right]+\frac{\partial}{\partial y}\left[\left(\frac{e}{2\omega}+\frac{1}{R}\right)\frac{\partial \omega^2}{\partial y}\right], \tag{2.16}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t}+\epsilon\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right) \tag{2.17}$$

and

$$U = \cos(x-t)+\epsilon[c_1+c_2\cos 2(x-t)]+O(\epsilon^2). \tag{2.18}$$

These equations and the boundary conditions

$$\left.\begin{aligned} &u\to U, \quad e\to 0, \quad \omega\to 0 \quad \text{as} \quad y\to\infty, \quad u=v=e=0 \quad \text{at} \quad y=0,\\ &\omega = \frac{1}{\beta}\left|\frac{\partial u}{\partial y}\right| S, \quad \tau = \frac{1}{\beta R}\frac{\partial u}{\partial y} \quad \text{at} \quad y=0 \end{aligned}\right\} \tag{2.19}$$

will be treated below for the case $R \gg 1$.

3. Analysis

As shown in previous analyses of turbulent boundary-layer flow (e.g. Bush & Fendell 1972), the boundary layer consists of two distinct regions, a defect layer in which viscous stresses can be neglected, and a wall layer in which the two stresses are of the same order of magnitude and in which diffusion of any variable F is large compared with $\mathrm{d}F/\mathrm{d}t$. When expressed in terms of our notation, the procedure suggested by Bush & Fendell consists of expanding the dependent variables in series of the form

$$F = \alpha_0(R)\,F_0(x, Y, t)+\alpha_1(R)\,F_1(x, Y, t)+\dots, \tag{3.1}$$

where the αs are gauge functions satisfying

$$\alpha_n(R) \ll \alpha_m(R) \quad \text{for} \quad m < n, \tag{3.2}$$

and where $Y = y$ in the treatment of the defect layer and $Y = Ry$ in the treatment of the wall layer. The solutions in the defect and wall layers can then be matched by using an intermediate variable. The calculation is less straightforward in the present study owing to complications connected with bottom roughness, and we prefer instead the following informal analysis.

To treat the wall layer, we introduce the variables u_w, v_w, E, Ω and η through

$$u_w = \frac{u}{\beta}, \quad v_w = \frac{R}{\beta}v, \quad E = e, \quad \Omega = \frac{\omega}{R}, \quad \eta = Ry, \tag{3.3}$$

substitute into (2.13)–(2.16), and neglect terms $O(1/\beta R)$. The resulting equations are

$$\frac{\partial u_w}{\partial x} + \frac{\partial v_w}{\partial \eta} = 0, \tag{3.4}$$

$$\left(1 + \frac{E}{\Omega}\right)\frac{\partial u_w}{\partial \eta} = \tau, \tag{3.5}$$

$$aE\left[\left|\frac{\partial u_w}{\partial \eta}\right| - \Omega\right] + \frac{\partial}{\partial \eta}\left[\left(1 + \frac{E}{2\Omega}\right)\frac{\partial E}{\partial \eta}\right] = 0, \tag{3.6}$$

$$\Omega^2\left[b\left|\frac{\partial u_w}{\partial \eta}\right| - c\Omega\right] + \frac{\partial}{\partial \eta}\left[\left(1 + \frac{E}{2\Omega}\right)\frac{\partial \Omega^2}{\partial \eta}\right] = 0, \tag{3.7}$$

together with the boundary conditions

$$u_w = v_w = E = 0, \quad \Omega = \left|\frac{\partial u_w}{\partial \eta}\right| S \quad \text{at} \quad \eta = 0. \tag{3.8}$$

Equations (3.4)–(3.8) are equivalent to a similar set of equations treated by Saffman & Wilcox, and from their analysis it can be inferred that the variables u, v, e and ω satisfy

$$u \to |\tau|^{\frac{1}{2}} \operatorname{sgn}\tau \left\{1 + \beta\left[\frac{1}{\kappa}\log\left(|\tau|^{\frac{1}{2}}y\right) + B\right]\right\}, \tag{3.9a}$$

$$v \to -y\frac{\partial}{\partial y}|\tau|^{\frac{1}{2}} \operatorname{sgn}\tau \left\{1 + \beta\left[\frac{1}{\kappa}\left(\log\left(|\tau|^{\frac{1}{2}}y\right) - 1\right) + B\right]\right\}, \tag{3.9b}$$

$$e \to |\tau|, \quad \omega \to \frac{|\tau|^{\frac{1}{2}}}{\kappa y} \tag{3.9c}$$

as $\eta \to \infty$, where B is a function of x, t and the roughness length, and where the asymptotic form given by (3.9) has been written in terms of the original boundary-layer coordinate y. Saffman & Wilcox also provide velocity profiles obtained by numerical integration of the wall-layer equations for various roughness lengths, but these are not needed in the present calculation. The key point here is that u asymptotes to the classical logarithmic law at the outer edge of the wall layer, and that the stress is constant through this layer.

The equations governing the defect layer are derived by noting that the large factor $1/\beta$ multiplying $|\partial u/\partial y|$ in (2.15) and (2.16) implies that u is independent of y in the limit $R \to \infty$. Therefore we introduce the variables u_d and v_d through

$$u = U + \beta u_d, \quad v = -y\frac{\partial U}{\partial x} + \beta v_d, \tag{3.10}$$

and obtain the defect-layer equations

$$\frac{\partial u_d}{\partial x} + \frac{\partial v_d}{\partial y} = 0, \tag{3.11}$$

$$\frac{\partial u_d}{\partial t} + \epsilon\left[\frac{\partial}{\partial x}(Uu_d) - y\frac{\partial U}{\partial x}\frac{\partial u_d}{\partial y}\right] + \epsilon\beta\left[u_d\frac{\partial u_d}{\partial x} + v_d\frac{\partial u_d}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{e}{\omega}\frac{\partial u_d}{\partial y}\right], \tag{3.12}$$

and two more equations obtained by substituting (3.10) into (2.15) and (2.16) and by omitting terms $O(1/R)$. The matching conditions imply that $v_d \to 0$ and that $U+\beta u_d$, e and ω agree with the expressions (3.9a, c) as $y \to 0$.

The matching condition

$$\frac{\partial u_d}{\partial y} \to \frac{|\tau|^{\frac{1}{2}} \operatorname{sgn} \tau}{\kappa y} \tag{3.13}$$

at the inner edge of the defect layer implies that

$$u_d \to |\tau|^{\frac{1}{2}} \operatorname{sgn} \tau \left[\frac{1}{\kappa} \log \left(|\tau|^{\frac{1}{2}} y\right) + D\right] \tag{3.14}$$

as $y \to 0$, where D is independent of y, and matching the resulting asymptotic form for u with (3.9a) yields the approximation

$$\tau = \frac{U|U|}{[1+\beta(B-D)]^2} \tag{3.15}$$

for the bottom stress. Therefore, if we let U^* and τ^* denote the dimensional versions of U and τ, τ^* can be expressed in terms of a drag coefficient c_f through

$$\tau^* = \tfrac{1}{2} c_f |U^*| \, U^*, \tag{3.16}$$

where

$$c_f = \frac{2\beta^2}{[1+\beta(B-D)]^2}. \tag{3.17}$$

The analysis by Bush & Fendell (1972) suggests solving the defect-layer equations by expanding the dependent variables in powers of β. This procedure is correct if the bottom is smooth, in which case B is an $O(1)$ constant and (3.12) and (3.17) can be approximated by neglecting terms $O(\beta)$ and $O(\beta^3)$ respectively. The resulting momentum equation is linear in the velocity defect u_d, and the drag coefficient is given by

$$c_f^{(s)} = 2\beta^2. \tag{3.18}$$

A similar set of approximations can be derived for flow over a rough bottom, but only if restrictions are placed on the magnitude of the Nikuradse equivalent sand roughness k_s.

If terms $O(\epsilon\beta)$ are neglected in the defect-layer equations, the approximate forms of the derivative of (3.12) with respect to y and of the equations for e and ω provide a closed set of equations for e, ω and $\partial u_d/\partial y$, with boundary conditions (3.9c) and (3.13) at $y=0$. It can be seen by inspection that D and u_d are $O(|\tau|^{\frac{1}{2}})$, and therefore the term βD in the denominator of (3.17) can be neglected if

$$\frac{\beta|U|}{(1+\beta B)^2} \ll 1. \tag{3.19}$$

It can also be shown that the neglect of terms nominally $O(\epsilon\beta)$ in the defect-layer equations is valid if (3.19) is satisfied.

Observations (Schlichting 1979, p. 620) indicate that

$$X = \beta\left\{8.5 + \frac{1}{\kappa}\log\left[\frac{\beta H}{|U|}(X-\beta D)\right]\right\} \tag{3.20}$$

for completely rough turbulent flow, where $X = 1+\beta B$ and where H is given in terms of k_s by

$$H = \frac{V}{\sigma k_s}. \tag{3.21}$$

Equation (3.20) can be expressed in the form $X = F(X)$ if the term βD is neglected, and evaluating the solution of this equation with U set equal to its peak value shows that the left-hand side of (3.19) is $O(10^{-1})$ or smaller for $H \geqslant 100$. For values of k_s such that this inequality is satisfied, (3.19) is satisfied with a reasonably good degree of accuracy, and the $O(\epsilon\beta)$ terms in (3.12) and the term βD in (3.15) and (3.17) can be neglected. Therefore, if X is now defined as the solution of

$$X = \beta\left[8.5+\frac{1}{\kappa}\log\frac{\beta H X}{|U|}\right], \tag{3.22}$$

the drag coefficient for completely rough turbulent flow can be approximated by

$$c_f^{(r)} = \frac{2\beta^2}{X^2}, \tag{3.23}$$

and (3.12) can be approximated by its linearized form.

In the ensuing analysis it will be assumed that the drag coefficient is given by $c_f = C_f$, where C_f denotes (3.18) or (3.23), and that (3.12) can be treated by omitting terms $O(\epsilon\beta)$. The argument given in the last few paragraphs shows that this set of approximations is consistent only if the sand roughness is sufficiently small, and therefore the results obtained below cannot be applied to the case of flow over extremely rough bottoms.

Assuming now that the bottom is smooth or that k_s is sufficiently small, we can determine the constant c_1 by integrating the approximated form of (3.12) from 0 to ∞. This yields the linearized Kármán integral equation (1.6) in the form

$$\frac{\partial f}{\partial t}+\epsilon\left[\frac{\partial}{\partial x}(Uf)+f\frac{\partial U}{\partial x}\right] = -\frac{C_f}{2\beta^2}|U|\,U, \tag{3.24}$$

where

$$f = \int_0^\infty u_d\,dy, \tag{3.25}$$

and using the fact that f is a periodic function of $\theta = x-t$ shows that (3.24) is equivalent to

$$\beta^2\frac{d}{d\theta}[(1-\epsilon U)^2 f] = \tfrac{1}{2}C_f|U|\,U(1-\epsilon U). \tag{3.26}$$

Integrating over a period then yields

$$\int_0^{2\pi}\tfrac{1}{2}C_f|U|\,U(1-\epsilon U)\,d\theta = 0, \tag{3.27}$$

which determines the constant c_1 in (1.1) and (1.3). Equation (3.27) is the dimensionless form of (1.5).

Up to this point no assumptions have been made regarding the magnitude of ϵ, and so (3.27) can be used to calculate the streaming motion associated with a periodic progressive wave of arbitrary magnitude. For the purposes of this study it suffices to restrict our attention to small-amplitude waves by assuming that $\epsilon \ll 1$, in which case the $O(\epsilon^2)$ term in (1.1) can be neglected and an analytic approximation can be derived for c_1 if C_f is assumed to be constant. The details of the derivation are unimportant, and we quote only the final result:

$$c_1 = \tfrac{1}{3}(1-c_2)+\epsilon^2[\tfrac{13}{81}+\tfrac{2}{27}c_2+\tfrac{32}{135}c_2^2+\tfrac{32}{81}c_2^3]+O(\epsilon^3) \tag{3.28}$$

The Stokes expansion is valid provided that $\epsilon c_2 \ll 1$, and for this parameter range substitution of (3.28) into (1.3) provides a useful approximation to the mass-transport

Wave	k (cm^{-1})	A (cm)	$Re \times 10^{-4}$	Q(theory)	Q(observed)
1	0.0418	4.3	0.113	0.648	0.50
3	0.0231	2.1	0.121	0.536	0.00
4	0.0231	4.3	0.505	0.538	0.44
5	0.0231	7.8	1.663	0.541	0.48
7	0.0160	4.5	1.035	0.350	0.32

TABLE 1. Theoretical and observed values of Q

$Re \times 10^{-6}$	$c_f \times 10^3$ (3.18)	$c_f \times 10^3$ (observed)
0.5	6.424	4.5
1.0	5.564	3.9
5.0	4.103	3.2
10.0	3.638	3.0

TABLE 2. Theoretical and observed values of c_f

velocity at the edge of the bottom boundary layer for flow over a smooth bottom. Sample calculations made using (3.23) show that (3.28) overestimates the value of c_1 in the case of flow over a rough bottom by about 20%.

The quantity

$$Q = \tfrac{4}{5}(c_1 + \tfrac{1}{2}) \tag{3.29}$$

is the ratio between the mass-transport velocity at the edge of the boundary layer predicted by the present theory and that given by Longuet-Higgins (1953). Values of Q were measured by Bijker *et al.* (1974) for waves on sloping bottoms and on the flat portion of the bottom in a wave flume. In the experiments the surface elevation was well predicted by Stokes' theory, and reflection from the end of the flume was negligible. The comparison shown in table 1 between the present theory and measurements is made using the observations by Bijker *et al.* on a flat part of the bottom of depth 45 cm adjacent to a beach with slope 1:25, and the theoretical value of Q was calculated using (3.28)

The figures from which the experimental values of Q were taken show a large amount of scatter for waves 1 and 3, and the computed values of the Reynolds number Re indicate that the flow in the bottom boundary layer for these cases may not have been fully turbulent. The closest degree of agreement between the calculated and observed values of Q is found for waves 5 and 7, for which the Reynolds number is largest and the amount of scatter the smallest. The present theory apparently overestimates the value of c_1, but the agreement with observations is satisfactory.

In table 2 a comparison is made between the theoretical value (3.18) of the drag coefficient and the observations made by Riedel *et al.* (1972) for turbulent flow over a smooth bottom. As anticipated, the discrepancy between theory and observations decreases with increasing Reynolds number.

In comparing (3.23) with measurements, it should be noted that the drag coefficients observed by Riedel *et al.* are expressed in terms of the maximum drag in a cycle, which occurs at values of x and t for which U is approximately equal to unity. For a Reynolds number $Re = 10^6$ and for roughness lengths such that $H \geqslant 300$, the drag coefficients obtained by setting $U = 1$ in (3.22) and (3.23) agree with observations with an error of less than 10%. In addition, the drag coefficient increases slowly with Re, in agreement with observations. The error is larger for smaller values of H, and reaches 23% for $Re = 10^6$ and $H = 100$. This suggests that the defect-layer equations must be solved numerically to calculate an accurate solution for flow over extremely rough bottoms.

In §1 we noted that the difference between observed values of U_{m} and theoretical values calculated assuming laminar flow increases with decreasing depth. Sample calculations show that this effect is predicted by the model used here and that U_{m} is negative for sufficiently small depth. The experimental finding that negative mass-transport velocities are observed under breaking waves cannot be confirmed on the basis of the present theory because of our inability to find a suitable representation for the bottom velocity under a breaking wave.

We conclude by discussing Taylor's (1919) hypothesis that 'The rate of dissipation of energy by friction is equal to the friction multiplied by the relative velocity of the surfaces between which the friction acts.' As applied to the present problem, Taylor's hypothesis implies that the rate of energy dissipation per unit horizontal area can be calculated by multiplying the bottom stress by the dimensional velocity U^* and averaging over a wavelength. Assuming that the drag coefficient is constant, and working to lowest order in ϵ, then yields

$$\Sigma = -\frac{2\rho}{3\pi} c_{\mathrm{f}} V^3, \tag{3.30}$$

where Σ is the rate of energy dissipation per unit area. Despite the apparent weakness of Taylor's reasoning, (3.30) is widely used in coastal engineering.

Equation (3.30) can be derived by noting that Σ is the negative of the average over a wavelength of the product of the dimensional vertical velocity v^* and pressure p^* evaluated at the edge of the boundary layer (Lamb 1932, pp. 8, 9). Here v^* is the flow due to displacement thickness, and is given by

$$v^* = \frac{\partial}{\partial x^*}(U^*\delta^*), \tag{3.31}$$

where δ^* is the displacement thickness. This equation can be expressed in the form

$$v^* = -\frac{\beta^2 V^2}{C} \frac{\mathrm{d}f}{\mathrm{d}\theta}, \tag{3.32}$$

where C is the phase velocity and f is defined by (3.25). The pressure p^* is given by irrotational theory in the form

$$p^* = \rho C U^* = \rho C V \cos\theta, \tag{3.33}$$

with an error $O(\epsilon)$, and with the same error (3.26) becomes

$$\frac{\mathrm{d}f}{\mathrm{d}\theta} = \frac{c_{\mathrm{f}}}{2\beta^2} |\cos\theta| \cos\theta. \tag{3.34}$$

Multiplying v^* by p^* and averaging over a wavelength then yields

$$\Sigma = -\tfrac{1}{2}\rho\, V^3 \langle c_{\mathrm{f}} |\cos^3\theta| \rangle, \tag{3.35}$$

and assuming that c_{f} is constant verifies Taylor's formula for the rate of energy dissipation.

4. Concluding remarks

As noted earlier, Longuet-Higgins's (1953) theory for the turbulent boundary layer under a progressive water wave overestimates the mass transport velocity at the edge of the layer by about a factor of two. Although his treatment differs considerably from ours, his equation relating the shear stress in the boundary layer to the vertical and free-stream velocities (Russell & Osorio 1958, p. 192, equation (22)) can be derived using our model by omitting terms $O(1/R)$ in (2.13) and (2.14) and by expanding the velocity components in powers of ϵ. At this point in his analysis Longuet-Higgins makes the simplifying assumption that the eddy viscosity is constant following a particle. This disagrees with Saffman's model and with other models for treating turbulent boundary layers, and can be shown to account for the differences between Longuet-Higgins's theoretical results and observations.

The results given in the present paper can be obtained by invoking only two assumptions, that the eddy viscosity and tangential velocity obey the classical law of the wall at the edge of the wall layer and that the tangential velocity in the defect layer is equal to the free-stream velocity plus a small perturbation. These features of the flow are predicted or assumed in all turbulence models known to the author, and therefore the low-order approximations for the mass-transport velocity and the drag coefficient obtained above are independent of the model. Higher-order approximations for these quantities would depend on the modelling assumptions, and might indicate which of the many turbulence models now in use provides the best results for oscillatory boundary layers.

This work was supported by the National Oceanographic and Atmospheric Administration under NOAA Grant 04-7-022-44020.

REFERENCES

Bijker, E. W., Kalwijk, J. P. T. & Pieters, T. 1974 In *Proc. 14th Coastal Engng Conf.*, p. 447. ASCE.

Brebner, A., Askew, J. A. & Law, S. W. 1966 In *Proc. 10th Coastal Engng Conf.*, p. 175. ASCE.

Bush, W. B. & Fendell, F. E. 1972 *J. Fluid Mech.* **56**, 657.

Collins, J. I. 1963 *J. Geophys. Res.* **68**, 6007.

Lamb, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

Lighthill, J. 1978 *Waves in Fluids*. Cambridge University Press.

Longuet-Higgins, M. S. 1953 *Phil. Trans. R. Soc. Lond.* A **245**, 66.

Riedel, H. P., Kamphuis, J. W. & Brebner, A. 1972 In *Proc. 13th Coastal Engng Conf.*, p. 587. ASCE.

Russell, R. C. H. & Osorio, J. D. C. 1958 In *Proc. 6th Coastal Engng Conf.*, p. 171. ASCE.

Saffman, P. G. 1970 *Proc. R. Soc. Lond.* A **317**, 417.

Saffman, P. G. 1974 *Stud. Appl. Maths* **53**, 17.

Saffman, P. G. & Wilcox, D. C. 1974 *AIAA J.* **12**, 541.

Schlichting, H. 1979 *Boundary-Layer Theory*, 7th edn. McGraw-Hill.

Taylor, G. I. 1919 *Phil. Trans. R. Soc. Lond.* A **220**, 1.